Year 2023-2024

The Chinese University of Hong Kong

Department of Mathematics

MMAT 5340 Probability and Stochastic Analysis

Homework 5: Martingale Convergence and Stopping Times

Due Date: 23:59 pm on Sunday, February 25th, 2024.

Please submit your homework on Blackboard

1. Let $(\xi_k)_{k\geq 1}$ be a sequence of independent and identically distributed random variables with standard Gaussian distribution, i.e. $\xi_k \sim \mathcal{N}(0,1)$. We define $X = (X_n)_{n\geq 0}$ as follows:
$$X_0 := 0,\ X_n := \sum_{k=1}^{n} \frac{1}{k}\xi_k,\ \text{for all } n \geq 1.$$
   (a) Prove that X is a martingale.
   (b) Prove that
$$\sup_{n\in\mathbb{N}} \mathbb{E}\big[|X_n|^2\big] < \infty.$$
   Hint: You may use the fact that
$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$
   without proof.
   (c) By the convergence theorem of the martingale (Theorem 2.4), we know that $X_n \to X_\infty$ a.s. and in L^2 for some random variable X_∞ as $n \to \infty$.
      i. Compute the characteristic function ψ_n of X_n, where ψ_n is defined as
$$\psi_n(\theta) := \mathbb{E}\big[e^{i\theta X_n}\big],\ \theta \in \mathbb{R}.$$
      ii. Compute
$$\psi(\theta) := \lim_{n\to\infty} \psi_n(\theta),\ \theta \in \mathbb{R}.$$
      iii. Identify the distribution of X_∞.
      Hint: ψ is the characteristic function of X_∞ and the distribution of a random variable is uniquely determined by its characteristic function.

2. Let $(\xi_k)_{k\geq 1}$ be a sequence of independent and identically distributed random variables such that $\mathbb{P}[\xi_k = \pm 1] = \frac{1}{2}$. We define $X = (X_n)_{n\geq 0}$ as follows:

$$X_0 := 0,$$
$$X_n := \sum_{k=1}^{n} 2^{k-1}\xi_k \mathbb{1}_{\{k\leq \tau\}}, \text{ where } \tau := \inf\{k \in \mathbb{N} : \xi_k = 1\}.$$

(a) Prove that X is a martingale.

(b) Compute $\mathbb{P}[\tau > n]$ and deduce that $\mathbb{P}[\tau < +\infty] = 1$

Hint: $\{\tau > n\} = \{\xi_1 = \cdots = \xi_n = -1\}$.

(c) Prove that $X_\tau = 1$ a.s.

Remark: It may be worth noting that in this case, we have

$$1 = \mathbb{E}[X_\tau] \neq \mathbb{E}[X_0] = 0.$$

(d) Compute $\mathbb{E}[|X_n|]$ and prove that

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|] < \infty, \quad \text{and} \quad \lim_{n\to\infty} X_n = X_\tau \text{ a.s.}$$

Hint: $\mathbb{E}\big[|X_n|\big] = \mathbb{E}\big[|X_n|\mathbb{1}_{\{\tau>n\}}\big] + \mathbb{E}\big[|X_\tau|\mathbb{1}_{\{\tau\leq n\}}\big]$.